

A Simple Quantum Cosmology

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Abstract

A simple and surprisingly realistic model of the origin of the universe can be developed using the Friedmann equation from general relativity, elementary quantum mechanics, and the experimental values of \hbar, c, G and the proton mass m_p . The model assumes there are N space dimensions (with $N > 6$), and the potential constraining the radius r of the invisible $N - 3$ compact dimensions varies as r^4 . In this model, the universe has zero total energy and is created from nothing. There is no initial singularity. If space-time is eleven dimensional, as required by M-theory, the scalar field corresponding to the size of the compact dimensions inflates the universe by about 26 orders of magnitude (60 e -folds). If $H_0 = 65 \text{ km sec}^{-1} \text{ Mpc}^{-1}$, the energy density of the scalar field after inflation results in $\Omega_\Lambda = 0.68$, in agreement with recent astrophysical observations.

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1 Introduction

Analysis of cosmic microwave and X-ray background radiation, extra-galactic radio sources, and Lyman- α lines from neutral hydrogen in the universe indicates the three large-scale space dimensions of our universe are isotropic and homogeneous [1]. The relevant variable in a homogeneous isotropic universe is the scale factor [2]. For a closed universe, this scale factor is the radius of curvature of the universe.

Aleksandr Friedmann used Einstein's general relativity to obtain the Friedmann equation for the scale factor R of a homogeneous isotropic universe [2]:

$$\left(\frac{dR}{dt}\right)^2 - \left(\frac{8\pi G}{3}\right)\varepsilon\left(\frac{R}{c}\right)^2 = -kc^2 \quad (1)$$

where ε is the energy density of the universe, the gravitational constant $G = 6.67 \times 10^{-8}$ cm³/g sec², and $c = 3 \times 10^{10}$ cm/sec. For a closed universe, $k = 1$; for a flat universe, $k = 0$; and for an open universe, $k = -1$. At present, the energy density is [3]

$$\varepsilon = \varepsilon_r \left(\frac{R_0}{R}\right)^4 + \varepsilon_m \left(\frac{R_0}{R}\right)^3 + \varepsilon_\phi,$$

where $\varepsilon_r, \varepsilon_m$, and ε_ϕ are, respectively, today's values of the radiation, matter and scalar field energy densities, and R_0 is the scale factor of the universe today. Multiplied by $\frac{1}{2}m$, equation (1) describes the motion of a fictitious particle with mass m and energy $-\frac{1}{2}kmc^2$ in the potential

$$V_R = -\frac{m}{2} \left(\frac{8\pi G}{3}\right) \left[\varepsilon_r \left(\frac{R_0}{R}\right)^4 + \varepsilon_m \left(\frac{R_0}{R}\right)^3 + \varepsilon_\phi \right] \left(\frac{R}{c}\right)^2.$$

Since $\varepsilon_r \approx 10^{-34}$ g/cm³ c², $\varepsilon_m \approx 10^{-29}$ g/cm³ c², and $R_0 \approx 10^{28}$ cm, the radiation term in the potential is approximately $-5.6 \frac{m}{2} \frac{10^{71}}{R^2}$ g (cm/sec)² and the matter term in the potential is about $-5.6 \frac{m}{2} \frac{10^{48}}{R}$. In the early universe, when $R \ll 10^{-5} R_0$, the radiation term dominated, so this paper neglects the matter term.

According to the Friedmann equation, the universe began with the "Big Bang," when $R = 0$ and the energy density was infinite. However, the occurrence of infinities in a physical theory can signal a breakdown in the theory [4]. Because general relativity is not believed to be valid at distances less than the Planck length, it has been hoped that a quantum mechanical approach would avoid the problem of the initial singularity in the Friedmann universe.

Besides the initial singularity, there are other problems with many cosmologies based on the Friedmann equation. For example, the original Friedmann cosmology can't explain why the universe

- is nearly flat (or, in other words, why it is so large that it appears to be nearly flat),

- is so homogeneous, in that regions that could never have communicated with each other by signals travelling at the speed of light have the same matter/energy distribution, and
- has such a smooth distribution of matter.

These problems (the flatness, horizon and smoothness problems) [2] are solved in inflationary models, where the universe contains a scalar field. Then, the universe can expand faster than the speed of light in an exponential expansion that makes the universe nearly flat, smooth and homogeneous. Ideally, a quantum mechanical approach to cosmology should also describe this inflationary phase.

The usual approach to quantum cosmology, involving the Wheeler-De Witt equation, is fraught with difficulties [5]. However, the canonical Hamiltonian quantization of standard cosmology by Elbaz *et al* [6] and Novello *et al* [7] avoids the complexities of the Wheeler-De Witt equation. For example, they show that the quantum dynamics of a closed homogeneous and isotropic radiation-dominated universe is equivalent to the quantum dynamics of a particle moving in one dimension in a potential

$$V(q) = -\frac{b^2}{4q^2},$$

where q is proportional to the radius of curvature of the universe and b is a constant.

The Schrödinger equation corresponding to equation (1) (which is the same as the Schrödinger equation of Elbaz *et al* [6] and Novello *et al* [7]) for a closed radiation-dominated Friedmann universe) can be written as

$$-\frac{\hbar^2}{2m} \frac{d^2}{dR^2} \psi - \frac{4\pi mG}{3c^2} \frac{\varepsilon_r R_0^4}{R^2} \psi = -\frac{\hbar^2}{2m} \frac{d^2}{dR^2} \psi - \frac{mG}{2} \frac{A}{R^2} \psi = -\frac{mc^2}{2} \psi \quad (2)$$

The solution to equation (2) can be expressed as

$$\psi = C_1 R h_{ip-\frac{1}{2}}(i\kappa R),$$

where h is the spherical Hankel function [8], or

$$\psi = C_2 \kappa \sqrt{R} K_{ip}(\kappa R),$$

where K is the modified Bessel function of the second kind [7]. Here, $\kappa = mc/\hbar$, the dimensionless index of the Bessel or Hankel function $p = \sqrt{\frac{m^2 G A}{\hbar^2} - \frac{1}{4}}$, and C_1 and C_2 are normalization constants. This stationary state wave function is zero at $R = 0$.

A time-dependent wave function describing the evolution of a closed radiation-dominated Friedmann universe must be a wave packet centered on the “effective energy” $-\frac{1}{2}mc^2$. Any superposition of stationary state solutions to equation (2) is also zero at $R = 0$. If general relativity were assumed valid at distances less than the Planck length, the Schrödinger

equation of Elbaz *et al* would be valid at these distances. So, quantum mechanics forbids initial (and final) singularities in a closed radiation-dominated Friedmann universe. Since the eigensolutions of the Schrödinger equations for flat and open radiation-dominated Friedmann universes are also zero at $R = 0$, this conclusion can be extended to flat and open radiation-dominated Friedmann universes. This analysis indicates the need for a model of the universe where the radius of curvature is never zero. One possibility is a quantum mechanical model allowing the universe to begin with a non-zero radius, by a quantum fluctuation from nothing.

Theories that adequately describe the four forces governing the universe (gravity, electromagnetism, the weak force, and the strong nuclear force) seem to require more than four space-time dimensions. If the extra dimensions are compact (for example, curled up so tightly that their characteristic size is the Planck length 10^{-33} cm), they are invisible under ordinary circumstances. These extra dimensions provide another degree of freedom allowing development of the simple model [9, 10] for the origin of the universe discussed in this paper. In addition, the size of the compact dimensions is a scalar field in the ordinary three dimensional space described by the Friedmann equation, and this scalar field produces the inflation needed to solve the problems of the earlier Big Bang cosmologies.

2 Overview of the Model

Assumptions necessary to develop the model

- The evolution of a homogeneous universe with N space dimensions can be described by a Schrödinger wave equation in an abstract N -dimensional Euclidean curvature space. The coordinate of the universe in each of the N curvature space dimensions is the radius of curvature of the corresponding dimension in the N -dimensional homogeneous physical space.
- The universe arises by a quantum fluctuation from nothing, so all total quantum numbers in the curvature space (and in the physical space) must be zero.
- The potential in the curvature space describing our universe has two terms. One term involves only the radius of curvature of the three space dimensions of a closed Friedmann universe. The other term involves only the characteristic radius of curvature of the $N - 3$ compact dimensions. So, the N -dimensional Schrödinger equation separates into two equations describing two distinct subspaces of the N -dimensional curvature space. One subspace corresponds to our homogeneous and isotropic Friedmann universe, and one subspace corresponds to the compact dimensions. The two subspaces do not exchange energy, except when the curvature energy of the compact dimensions

drops to the ground state, injecting energy and entropy into the Friedmann dimensions and causing inflation

- The curvature of the Friedmann space is described by the quantum mechanical analogue of equation (1). This is the s-wave Schrödinger equation justified by Elbaz *et al* [6] and Novello *et al* [7]. Since the total angular momentum in curvature space must be zero, an s-wave Schrödinger equation must also be used to describe the evolution of the compact space.
- In the initial state of the universe, the radius of curvature of all of the dimensions was some small multiple a of the Planck length, and it was not changing. In this initial state, gravity and the strong-electroweak force had the same strength, and the Planck length was about nineteen orders of magnitude larger than it is today. Today, the characteristic radius of curvature of the compact dimensions is a times the Planck length.

Inputs needed for numerical estimates

- The number of space-time dimensions: M-theory seems to be the leading candidate for the fundamental theory of the four forces governing the universe, so this paper assumes space-time is eleven dimensional and $N = 10$.
- The long-range behavior of the effective potential constraining the size of the compact dimensions: One can assume $V_r = k_n r^4$ to get a universe similar to our own [9, 10]. Or, one can assume the energy spectrum of the compact dimensions must be that necessary for the Tseytlin-Vafa [11] dimensional collapse scenario in string theory. In the latter case, it can be shown that the WKB approximation for the energy spectrum requires $V_r = k_n r^4$.
- The experimental values of four fundamental physical constants: the proton mass $m_p = 1.67 \times 10^{-24}$ g, $\hbar = 1.05 \times 10^{-27}$ g cm²/sec, c , and G .

3 Model Development

If the universe was created by a quantum fluctuation from nothing, it must be closed, with all quantum numbers (including the total energy) equal to zero. If it is homogeneous, the relevant variables are the radii of curvature of each of the dimensions. Consequently, an $(N + 1)$ -dimensional quantum mechanical model of the origin of the universe can be developed [9, 10], where space is initially an N -dimensional sphere. The model is formulated in an N -dimensional Euclidean curvature space (with $N > 6$) describing the curvature of a

homogeneous N -dimensional physical space. In the simplest case, today's curvature space has subspaces related to the Friedmann universe and the $N - 3$ compact dimensions, and the curvature of all the compact dimensions is the same. The coordinate in each dimension of a state in the curvature space is the radius of curvature of the corresponding dimension of that state in the N -dimensional physical space.

When the total energy and total angular momentum in curvature space are zero, the Schrödinger equation for the N -dimensional radius of curvature is

$$-\frac{\hbar^2}{2m}\nabla_{\mathfrak{R}}^2\Psi + V_{\mathfrak{R}}\Psi = 0,$$

where \mathfrak{R} is the magnitude of an N -dimensional vector $\vec{\mathfrak{R}}$ and m is an effective mass. Today, the “gravitational structure constant” $\frac{Gm_p^2}{\hbar c} = 5.91 \times 10^{-39}$ is the ratio of the strength of gravity to the strength of the strong force, the Planck mass $M = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-5}$ g, and the Planck length $\delta = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \times 10^{-33}$ cm. Initially, gravity and the strong-electro-weak (SEW) force had equal strength, and $\frac{G_im_p^2}{\hbar c} = 1$. The gravitational constant was initially $G_i = \left(\frac{M}{m_p}\right)^2 G = 1.70 \times 10^{38} G$, so $\frac{G_im_p^2}{\hbar c} = \frac{GM^2}{\hbar c} = 1$. The Planck length was $\delta_i = \sqrt{\frac{\hbar G_i}{c^3}} = \left(\frac{M}{m_p}\right) \delta = 2.11 \times 10^{-14}$ cm and the Planck mass was $M_i = \sqrt{\frac{\hbar c}{G_i}} = m_p$.

In this model, the universe can be envisioned to begin by a quantum fluctuation from nothing into a spherically symmetric N -dimensional universe with zero total energy, $\sqrt{\langle\mathfrak{R}^2\rangle} = a\delta_i$ and $\dot{\mathfrak{R}} = 0$. The fundamental length in this curved space is the circumference of the N -sphere. If $a = \frac{1}{2\pi}$, this fundamental length is δ_i . In this initial state, none of the N space dimensions were distinguishable, and the only length scale is δ_i because all forces were initially equal. Such an initial state might be an unbroken symmetry state of some fundamental theory of the four forces governing the universe.

After the symmetric initial state arose from nothing by a quantum fluctuation, a quantum tunneling transition occurred from the initial state to another state with zero total curvature energy, where $\mathfrak{R}^2 = R^2 + r^2$, R is the radial coordinate in the three dimensional subspace describing the curvature of the isotropic Friedmann universe, and r is the radial coordinate in the $n = N - 3$ dimensional subspace describing the curvature of the compact dimensions. At the transition, $\sqrt{\langle R^2 \rangle} = \sqrt{\langle r^2 \rangle} = a\delta_i$ and $\dot{R} = \dot{r} = 0$. This post-transition state was the beginning of today's universe, where the size of the compact dimensions corresponds to a gauge singlet scalar field ϕ [12] that is constant throughout the Friedmann universe and drives inflation.

Notice that the universe could arise directly from a quantum fluctuation from nothing into the post-transition stage described above. However, the assumed transition from a completely symmetrical initial state allows a connection to more fundamental theories of the forces controlling the universe.

The precise relation between the gauge singlet scalar field ϕ and the characteristic size of the compact dimensions is not important in this simple model. One realization of the model might involve a scalar field related to the size of the compact dimensions by $\phi = \frac{\zeta}{a\delta_i} \sqrt{\frac{\hbar}{c}} \ln\left(\frac{a\delta_i}{r}\right)$. Then, the scalar field $\phi = 0$ when $r = a\delta_i$. The value of the real number ζ would have to be obtained from a more fundamental theory of the four forces governing the universe, such as M-theory.

The model assumes $V_{\mathcal{R}} = V_R + V_r$, so $\Psi = \Psi(R)\Psi(r)$ and

$$\left[\frac{1}{\Psi(R)} \frac{-\hbar^2}{2m} \nabla_R^2 \Psi(R) + V_R \right] + \left[\frac{1}{\Psi(r)} \frac{-\hbar^2}{2m} \nabla_r^2 \Psi(r) + V_r \right] = 0,$$

where each bracket is a constant, denoted $-E$ and E respectively. So, the curvature energy of the closed Friedmann universe is less than zero, the curvature energy of the extra dimensions is greater than zero, and the total curvature energy of the universe is zero. In this model, the universe evolved from an excited state with large curvature energies $-E$ and E (where $|E| \gg \frac{1}{2}mc^2$) reached by a quantum transition from the spherically symmetric, zero curvature energy initial state with indistinguishable space dimensions. Today's universe is a quantum state where the curvature energy of the Friedmann universe has the Einstein value $-\frac{1}{2}mc^2$, and the compact dimensions are in the ground state of the potential V_r , with curvature energy $\frac{1}{2}mc^2$. The resulting simple quantum mechanical model for the origin of the universe is schematically outlined in Figure 1 (the details of Figure 1 will be explained below). The model provides a quantum theory of space, but it is certainly not the long-sought quantum field theory of gravity that will truly unify quantum mechanics and general relativity.

In the model, the Schrödinger equation for $\psi(R)$ is the quantum analog of the Friedmann equation for a universe containing radiation and a scalar field:

$$\begin{aligned} -\frac{\hbar^2}{2m_\phi} \frac{d^2}{dR^2} \psi - \frac{4\pi m_\phi G_\phi}{3} (\varepsilon_r + \varepsilon_\phi) \left(\frac{R}{c}\right)^2 \psi &= -E' \psi \quad \text{or} \\ -\frac{\hbar^2}{2m_\phi} \frac{d^2}{dR^2} \psi - \frac{4\pi\delta_\phi}{3} \left(\frac{A'}{R^2} + \varepsilon_\phi R^2\right) \psi &= -E' \psi \end{aligned} \quad (3)$$

where the scalar field energy density $\varepsilon_\phi = \dot{\phi}^2 + V_\phi(\phi)$ models the effect of the compact dimensions on the Friedmann universe. The subscript ϕ indicates that G , the effective mass, the Planck length, and the scalar field energy density depend on the value of the scalar field ϕ . If $R \rightarrow 0$ or $\varepsilon_r \gg \varepsilon_\phi$, equation (3) reduces to the Schrödinger equation (2) for a radiation dominated universe [6, 7]. An s-wave Schrödinger equation must be used for the compact dimensions to make the total N-dimensional “angular momentum” in the curvature space zero. Writing $\Psi = R^{-1}\psi(R)r^{-(n-1)/2}\psi'(r)$, the separated Schrödinger

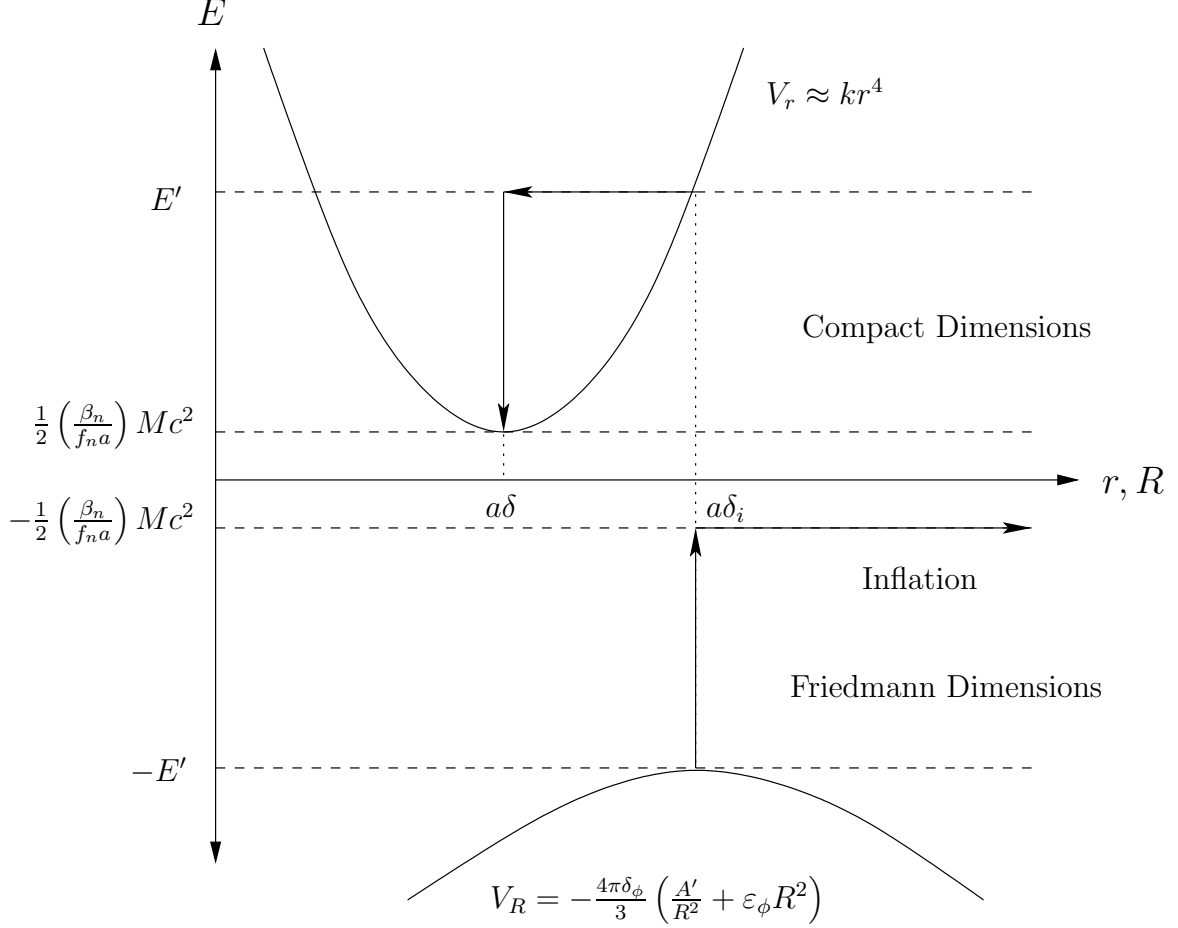


Figure 1: Schematic sketch of the initiation of inflation. (Not drawn to scale). Here, $E' = \frac{(n-1)(n-3)Mc^2}{16\beta_n f_n^5 a} \left(\frac{\delta_i}{\delta} \right)^4$.

equation becomes

$$\left[\frac{1}{\psi(R)} \frac{-\hbar^2}{2m} \frac{d^2\psi(R)}{dR^2} + V_R \right] + \left[\frac{1}{\psi'(r)} \frac{-\hbar^2}{2m} \frac{d^2\psi'(r)}{dr^2} + \left(\frac{\hbar^2(n-1)(n-3)}{8mr^2} + V_r \right) \right] = 0.$$

If $V_r = k_n r^4$, the model produces a universe like our own [8, 9] if the minimum of the effective potential $\frac{\hbar^2(n-1)(n-3)}{8mr^2} + k_n r^4$ in the compact dimensions is near $r = a\delta$. Specifically, if the minimum of the effective potential in the compact dimensions is at $r = f_n a\delta$, $k_n = \frac{\hbar^2(n-1)(n-3)}{16mf_n^6 a^5 \delta^6}$. The factor f_n , calculated below, ensures that the radius of the compact dimensions today is $\sqrt{\langle r^2 \rangle} = a\delta$. Approximating the effective potential for the compact dimensions by a harmonic oscillator potential near its minimum, the ground state energy of the compact dimensions is $E_g = \frac{\hbar^2 \beta_n^2}{2mf_n^2 a^5 \delta^6}$, where $\beta_n^2 = \frac{3}{8}(n-1)(n-3) + \sqrt{\frac{3}{2}(n-1)(n-3)}$. Setting $E_g = \frac{1}{2}mc^2$ establishes the effective mass as $m = \left(\frac{\beta_n}{f_n a} \right) M$, so $k_n = \frac{\hbar^2(n-1)(n-3)}{16\beta_n M f_n^5 a^5 \delta^6} =$

$\frac{(n-1)(n-3)Mc^2}{16\beta_n f_n^5 a^5 \delta^4}$. The ground state wavefunction of the compact dimensions is

$$\psi'(r) \approx \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\gamma_n}{f_n a \delta}} \exp\left(-\frac{\gamma_n^2 (r - f_n a \delta)^2}{f_n^2 a^2 \delta^2}\right),$$

where $\gamma_n^2 = \sqrt{\frac{3(n-1)(n-3)}{8}}$. Setting $x = r - f_n a \delta$,

$$\begin{aligned} \langle r^2 \rangle &\approx \left\{ \int_{-f_n a \delta}^0 (x + f_n a \delta)^2 \exp\left[-\left(\frac{\gamma_n x}{f_n a \delta}\right)^2\right] dx + \int_0^\infty (x + f_n a \delta)^2 \exp\left[-\left(\frac{\gamma_n x}{f_n a \delta}\right)^2\right] dx \right\} \\ &\times \left\{ \int_{-f_n a \delta}^0 \exp\left[-\left(\frac{\gamma_n x}{f_n a \delta}\right)^2\right] dx + \int_0^\infty \exp\left[-\left(\frac{\gamma_n x}{f_n a \delta}\right)^2\right] dx \right\}^{-1} \\ &= \frac{f_n^2 a^2 \delta^2 \left\{ -\frac{\exp(-\gamma_n^2)}{\gamma_n} + \sqrt{\pi} [1 + \text{Erf}(\gamma_n)] \left(1 + \frac{1}{2\gamma_n^2}\right) \right\}}{\{\sqrt{\pi} [1 + \text{Erf}(\gamma_n)]\}}, \end{aligned}$$

where $\text{Erf}(\gamma_n)$ is the error function of γ_n . So, $\sqrt{\langle r^2 \rangle} = a\delta$ if

$$f_n = \left\{ \frac{-\frac{\exp(-\gamma_n^2)}{\gamma_n} + \sqrt{\pi} [1 + \text{Erf}(\gamma_n)] \left(1 + \frac{1}{2\gamma_n^2}\right)}{\sqrt{\pi} [1 + \text{Erf}(\gamma_n)]} \right\}^{-\frac{1}{2}}.$$

After the quantum transition from the initial state, and just prior to inflation, the universe was still in a symmetric state with radius $a\delta_i$, as indicated in Figure 1. Immediately after the transition, the compact dimensions were in a highly excited state of the effective potential V_r , with wave packet localized at the classical turning radius $r = a\delta_i$, and curvature energy $E' = \frac{(n-1)(n-3)Mc^2}{16\beta_n f_n^5 a} \left(\frac{\delta_i}{\delta}\right)^4$. The curvature energy in the Friedmann dimensions at transition, $-E'$, coincided with the top of the effective potential in equation (3), at $R_{\text{peak}} = \delta_i$, where $R_{\text{peak}}^4 = \frac{A'}{\varepsilon_\phi}$. So the transition resulted in a state with wave packet centered at $R = a\delta_i$, in unstable equilibrium, with $\dot{R} = 0$. At transition $\varepsilon_\phi = \frac{A'}{a^4 \delta_i^4}$, so $\varepsilon_r = \varepsilon_\phi$ and $\frac{8\pi m_\phi G_\phi A'}{3c^2 a^2 \delta_i^2} = \frac{(n-1)(n-3)Mc^2}{16\beta_n f_n^5 a} \left(\frac{\delta_i}{\delta}\right)^4$. Incidentally, the effective r^3 force constraining the size of the compact dimensions in this model is related to the effective $1/R^3$ radiation force in the Friedmann universe by the replacement $R \rightarrow 2^{-2/3} a^2 \delta_i^2 / r$.

When the curvature energy of the compact dimensions dropped to the ground state energy $\frac{1}{2} \left(\frac{\beta_n}{f_n a} Mc^2\right)$, the curvature energy of the Friedmann universe was raised to the Einstein value $-\frac{1}{2} \left(\frac{\beta_n}{f_n a} Mc^2\right)$. The scalar field ϕ changed from its initial value ϕ_i to its present value ϕ_f as the characteristic size of the compact dimensions decreased from δ_i to δ , G decreased from G_i to its present value, and the Planck mass increased from m_p to its present value $M = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-5}$ g.

In this model, inflation occurred when the characteristic size of the compact dimensions shrank from δ_i to δ and the curvature energy of the compact dimensions dropped from the transition energy E' to the ground state energy E_g , raising the curvature energy of the

Friedmann dimensions to $-E_g$. Entropy was injected into the scalar field in the Friedmann dimensions, and transferred to radiation as the scalar field decayed during inflation. When $R > R_{\text{peak}} = a\delta_i$, the ε_ϕ term in equation (3) dominated and the Friedmann universe inflated. Figure 1 schematically indicates the process that initiated inflation.

Since $\dot{r} = 0$ at transition, $\dot{\phi} = 0$ and the scalar field energy density at transition was $\varepsilon_\phi = \dot{\phi}^2 + V_\phi(\phi_i) = V_\phi(\phi_i)$. When the compact dimensions reached their ground state at the end of inflation, $\langle \dot{r}^2 \rangle = \langle \dot{\phi}^2 \rangle = 0$ thereafter, and the scalar field energy density remained constant at $\varepsilon_\phi = \dot{\phi}^2 + V_\phi(\phi_f) = V_\phi(\phi_f)$. This is consistent with the observation that, after the scalar field energy stopped decaying to radiation at the end of inflation, the energy conservation equation can be separated into two parts, one for radiation and one for the scalar field [3, pg. 727]. Then the scalar field pressure is $p_\phi = \dot{\phi}^2 - V_\phi(\phi) = -V_\phi(\phi_f) = -\varepsilon_\phi$, and the scalar field energy density is constant thereafter. So, at the end of inflation, after the scalar field stopped decaying to radiation, $\varepsilon_\phi \ll \varepsilon_{\text{rad}}$, and the radiation-dominated universe satisfied equation (2) with

$$A = \frac{(n-1)(n-3)a^2 \hbar}{8\beta_n^2 f_n^4} \frac{1}{c} \left(\frac{\delta_i}{\delta} \right)^6. \quad (4)$$

4 Numerical Estimates

M-theory (involving eleven-dimensional space-time) seems to be the leading candidate for the theory of the four forces governing the universe, so the remainder of this paper assumes $N = 10$, $n = 7$ and $a = 1/2\pi$. Then, $\beta_7^2 = 15$, $\gamma_7 = 3$, and $f_7 = 0.93$. From equations (2) and (4), $V_{\text{radiation}} = 7.8 \frac{m}{2} \frac{10^{67}}{R^2} \text{ g cm}^2/\text{sec}^2$, and the calculated radiation density is $\varepsilon_r = 1.4 \times 10^{-38} \text{ g/cm}^3 \text{c}^2$ if $R_0 = 10^{28} \text{ cm}$. However, the model does not explicitly account for strong-electroweak symmetry breaking. Nuclear energy levels, determined by the strong force, are roughly 10^6 times the atomic energy levels determined by the electromagnetic force, indicating that strong-electroweak symmetry breaking would increase the initial radiation density of the universe by up to six orders of magnitude. When some of this primordial radiation decays to hadrons, the remaining photons could then result in today's microwave background radiation energy density of $\varepsilon_r = 4.66 \times 10^{-34} \text{ g/cm}^3 \text{c}^2$ [13].

The extent of inflation can be estimated by assuming the curvature energy of the compact dimensions dropped instantaneously to the ground state energy when the size of the compact dimensions reached $\langle r \rangle = a\delta$. The ratio of $\frac{E'}{E_g}$ the transition energy to the ground state energy is $\frac{E'}{E_g} = \frac{(n-1)(n-3)}{8\beta_n^2 f_n^4} \left(\frac{\delta_i}{\delta} \right)^4 = 7.73 \times 10^{75}$. At the end of inflation, the compact dimensions were in their ground state and could no longer transfer energy to the Friedmann dimensions. The scalar field had decayed to radiation and could no longer transfer entropy to the radiation field. Assuming the curvature energy of the compact dimensions

dropped instantaneously to the ground state when the size of the compact dimensions reached $\langle r \rangle = a\delta$ [10], the entropy of the compact dimensions was reduced by a factor of 0.13×10^{-75} and the entropy of the scalar field in the Friedmann universe was increased by a factor of 7.73×10^{75} at the beginning of inflation. Then, if T_0 was the temperature of the Friedmann universe at the end of inflation, when entropy in the scalar field stopped being transferred to radiation, the temperature of the scalar field at the beginning of inflation was $(7.73 \times 10^{75})^{1/3} T_0 = 1.98 \times 10^{25} T_0$. During isentropic expansion, $(RT)^3$ remains constant. So, the injection of entropy from the collapse of the compact dimensions increased the scale factor of the Friedmann universe by a factor of 1.98×10^{25} as the Friedmann universe expanded exponentially and isentropically (driven by the scalar field), until inflation ended when $\dot{\phi} \approx 0$ and the temperature of the universe was T_0 . This 58 e -fold inflation is within the limits set by fluctuations in the microwave background radiation [14]. If the strong-electroweak symmetry broke during inflation, as it must have to prevent monopole dominance, the temperature should increase by a factor of about $(10^6)^{1/3}$ at some instant during inflation. This will increase the inflation by a factor of about $(10^6)^{1/3} = e^{4.6}$, for a total inflation of more than 60 e -folds.

In this model, the vacuum energy density today is the scalar field energy density per unit coordinate volume (i.e., the vacuum energy density is the scalar field energy density per unit coordinate volume multiplied by the number of unit coordinate volumes in the universe and divided by the volume of the universe expressed in terms of unit coordinate volumes). The vacuum energy density has remained constant since the scalar field decoupled from the radiation field at the end of inflation. It is the energy density associated with the creation of 1 cm^3 of space during the continuing expansion of the universe.

The scalar field energy density in the co-moving volume of the universe at the beginning of inflation can be obtained from the Schrödinger equation (3). At the moment of transition, when $\dot{R} = 0$ and the scale factor $R = a\delta_i$, equation (3) becomes $\frac{4\pi\delta_i}{3} 2\varepsilon_\phi a^2 \delta_i^2 = E'$, because $\varepsilon_r = \varepsilon_\phi$ at transition. This scalar field energy density ε_ϕ is the total scalar field energy in the 3-sphere (with scale factor $R = a\delta_i$) comprising the initial Friedman universe, divided by the volume of the 3-sphere. It is related to η_ϕ , the scalar field energy density per unit coordinate volume at the beginning of inflation, by $\varepsilon_\phi = \frac{1}{2\pi^2 a^3 \delta_i^3} \eta_\phi$ [15]. So, the energy density per unit coordinate volume of the scalar field in the Friedmann universe at the start of inflation was

$$\eta_\phi = \left(\frac{3\pi}{4f_7^5} \right) 2.18 \times 10^{92} \text{ g cm}^{-1} \text{ sec}^{-2} = 7.38 \times 10^{92} \text{ g cm}^{-1} \text{ sec}^{-2}.$$

A spatially constant scalar field has only one degree of freedom, and the energy density in the scalar field is proportional to the fourth power of the temperature. So, the energy

density per unit coordinate volume of the scalar field at the end of inflation was

$$\eta_e = \frac{T_0^4}{(1.98 \times 10^{25} T_0)^4} 7.38 \times 10^{92} \text{ g cm}^{-1} \text{ sec}^{-2} = 4.80 \times 10^{-9} \text{ g cm}^{-1} \text{ sec}^{-2} = \varepsilon_e.$$

Primack [16] finds $H_0 = 65 \text{ km sec}^{-1} \text{ Mpc}^{-1}$, based on recent astrophysical measurements, so the critical density $\rho_c = \frac{3H_0^2}{8\pi G} = 7.91 \times 10^{-30} \text{ g cm}^{-3}$ and the critical energy density $\varepsilon_c = \rho_c c^2 = 7.11 \times 10^{-9} \text{ g cm}^2 \text{ sec}^{-2} \text{ cm}^{-3}$. Therefore, in this model, $\Omega_\Lambda = \frac{\varepsilon_e}{\varepsilon_c} = 0.68$, in agreement with the value $\Omega_\Lambda \approx 0.7$ obtained by Primack [16] from astrophysical data.

5 Conclusions

M-theory, a prime candidate for the fundamental theory of the forces governing the universe, requires eleven dimensional space-time with seven compact space dimensions. The model outlined above simulates two key features of M-theory as applied to cosmology: a scalar field in the Friedmann equation [17], and the Tseytlin-Vafa dimensional compactification scenario [11]. The vacuum energy density in the model can be identified with the cosmological constant/“dark energy” in the Friedmann equation for our four-dimensional universe that arises in M theory [17].

The Schrödinger equation for the effective potential $\frac{\hbar^2(n-1)(n-3)}{8mr^2} + k_n r^4$ in the compact dimensions cannot be solved exactly. However, the energy levels are related to the classical turning radius r_T by $E = k_n r_T^4$ and the WKB approximation shows that allowed values of r_T satisfy $r_T^3 \propto \left(j + \frac{1}{2}\right) \pi$, where j is an integer. This indicates that the energy spectrum varies as the radius of the compact dimensions if the effective potential varies as r^4 for large r .

The solutions to the Schrödinger equation (2) for a radiation dominated Friedmann universe involve the factor $p = \sqrt{\frac{8\pi m^2 G \varepsilon_r R_0^4}{3\hbar^2 c^2}} - \frac{1}{4}$. For $p^2 \gg 1$, the energy levels [8] of a radiation dominated Friedmann universe are $E_n = \frac{mc^2}{2} \exp\left(\frac{2\pi j'}{p}\right) \approx \frac{mc^2}{2} \left(1 + \frac{2\pi j'}{p}\right)$, where j' is an integer. The r.m.s. radius of the Friedmann dimensions is

$$\sqrt{\langle R^2 \rangle} = \left[\frac{\int_0^\infty R^3 K_{ip}(\kappa R) K_{-ip}(\kappa R) dR}{\int_0^\infty R K_{ip}(\kappa R) K_{-ip}(\kappa R) dR} \right]^{\frac{1}{2}} = \kappa^{-1} \sqrt{\frac{2(1+p^2)}{3}}.$$

Since $\kappa^{-1} = \frac{\hbar}{mc} = \frac{f_n a}{\beta_n} \delta$,

$$\sqrt{\langle R^2 \rangle} = \frac{f_n a}{\beta_n} \delta \sqrt{\frac{2(1+p^2)}{3}} \approx p \left(\frac{f_n a \delta}{\beta_n} \right) \sqrt{\frac{2}{3}},$$

so the energy spectrum goes as $1/R$.

Tseytlin and Vafa [11] claim that string winding modes in $N - 3$ dimensions, with an energy spectrum varying as r , compact those dimensions. The constricting effect of winding

modes can only be overcome in three of the space dimensions, where the energy spectrum of the momentum modes varies as δ_i/R , according to the T-duality of string theory. In the expanding three dimensions, the massless string momentum modes are the photons of the radiation-dominated Friedmann universe. So, the model simulates the dimensional compactification scenario in string theory envisioned by Tseytlin and Vafa. Furthermore, once the number of space dimensions is specified, and $V_r = k_n r^4$ is chosen to accommodate the Tseytlin/Vafa dimensional compactification mechanism, only the experimental values of \hbar, c, G and the proton mass m_p are needed to obtain numerical results from the model.

To conclude, a simple and surprisingly realistic $N + 1$ dimensional quantum mechanical model of the universe can be developed using the Friedmann equation from general relativity, elementary quantum mechanics, and measured values of \hbar, c, G , and the proton mass. The model suggests that the *details* of extra-dimensional collapse are less important than the *fact* of extra-dimensional collapse in explaining the inflation of our three-dimensional universe and the size of the cosmological constant/vacuum energy density.

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